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$N = 4$ supersymmetric mechanics in harmonic superspace

Evgeny Ivanov

Bogoliubov Laboratory of Theoretical Physics
JINR, 141980 Dubna, Russia
E-mail: eivanov@thsun1.jinr.ru

Olaf Lechtenfeld

Institut für Theoretische Physik, Universität Hannover
Appelstraße 2, 30167 Hannover, Germany
E-mail: lechtenf@itp.uni-hannover.de

Abstract: We define $N = 4$, $d = 1$ harmonic superspace $\mathbf{HR}^{1+2|4}$ with an SU(2)/U(1) harmonic part, SU(2) being one of two factors of the R-symmetry group SU(2) $\times$ SU(2) of $N = 4$, $d = 1$ Poincaré supersymmetry. We reformulate, in this new setting, the models of $N = 4$ supersymmetric quantum mechanics associated with the off-shell multiplets $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$. The latter admit a natural description as constrained superfields living in an analytic subspace of $\mathbf{HR}^{1+2|4}$. We construct the relevant superfield actions consisting of a sigma-model as well as a superpotential parts and demonstrate that the superpotentials can be written off shell in a manifestly $N = 4$ supersymmetric form only in the analytic superspace. The constraints implied by $N = 4$ supersymmetry for the component bosonic target-space metrics, scalar potentials and background one-forms automatically follow from the harmonic superspace description. The analytic superspace is shown to be closed under the most general $N = 4$, $d = 1$ superconformal group $D(2, 1; \alpha)$. We give its action on the analytic superfields comprising the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplets, reveal a surprising relation between the latter and present the corresponding superconformally invariant actions. The harmonic superspace approach suggests a natural generalization of these multiplets, with a $[\mathbf{2}(\mathbf{n}{+}\mathbf{1}), \mathbf{4n}, \mathbf{2}(\mathbf{n}{-}\mathbf{1})]$ off-shell content for $n > 2$.

Keywords: Extended Supersymmetry, Superspaces, Field Theories in Lower Dimensions.

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Contents

1. Introduction

Models of supersymmetric quantum mechanics (SQM) with extended $N{\geq}2$, $d = 1$ supersymmetry have plenty of uses (see a recent review [1]). For instance, they describe the low-energy dynamics of monopoles in $N = 4$ supersymmetric Yang-Mills theory [2]. Some variants of $N = 4$ superconformal mechanics [3] play the role of conformal field theories in the $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence and describe the near-horizon dynamics of black-hole solutions of supergravity. Supersymmetric extensions of integrable $d = 1$ models, e.g. Calogero-Moser type systems [4, 5], are expected to have interesting implications in string theory [6]. It is peculiar that not all $d = 1$ supersymmetric models can be directly recovered from appropriate $d > 1$ theories via dimensional reduction. They reveal some special target-space geometries which have no direct counterparts in higher dimensions (see e.g. [7, 8]). For $N = 4$, $d = 1$ models the just-mentioned peculiarity manifests itself, in particular, in the fact that the most general $N = 4$, $d = 1$ superconformal symmetry is provided by the supergroup $D(2, 1; \alpha)$ [9]–[11] which only for special values of the parameter α is isomorphic to $\mathrm{SU}(1, 1|2)$ obtainable from higher-dimensional superconformal groups by dimensional reduction.

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In many studies of $d = 1$ supersymmetry (see e.g. [7, 9]) the $d = 1$ actions invariant under extended supersymmetries are constructed in components or/and in $N = 1$ superfields, proceeding from the most general $N = 1$ supersymmetric form of such actions and revealing the restrictions which are imposed on the relevant target geometries by the requirement of invariance with respect to additional supersymmetries. In such formulations the higher supersymmetries are non-manifest and frequently on-shell. Like for the case of supersymmetry in $d > 1$, it is desirable to have formulations of $d = 1$ supersymmetric theories in the appropriate superspaces where *all* their underlying supersymmetries are off-shell and manifest. Then the constraints which ensure the relevant target-space geometries to be consistent with extended supersymmetry are valid á priori and, in fact, can be read off by studying the component structure of the action. For $N = 4$ supersymmetric mechanics such formulations have been pioneered in [12]–[14] and further elaborated e.g. in [11] and [15]–[19]. Until present, only the standard type of $N = 4$, $d = 1$ superspaces was utilized, namely the real (1|4)-dimensional and chiral (1|2)-dimensional superspaces $\mathbf{R}^{1|4}$ and $\mathbf{C}^{1|2}$. On the other hand, it is known that many remarkable geometric features of extended supersymmetric theories are manifest only in harmonic superspace formulations [20, 21]. One can expect that such formulations visualize the non-standard target-space geometries of $d = 1$ models like they visualize the hyper-Kähler and quaternion-Kähler geometries of $N = 2$, $d = 4$ supersymmetric sigma models [21] (and their dimensionally-reduced descendants). Also, a reformulation of $N = 4$ SQM models in harmonic superspace might help in constructing SQM models with more than four supersymmetries by joining models associated with different $N = 4$, $d = 1$ supermultiplets.

As a step toward these goals, in this paper we present the harmonic superspace formulation of the $N = 4$ supersymmetric mechanics model proposed and studied in [22]–[25] and [13, 11] and further elaborated for the $N = 4$ superconformally invariant case in [19]. It is associated with the off-shell supermultiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ comprising three physical and one auxiliary bosonic fields and four fermionic fields. Furthermore, we discuss along similar lines a model based on a different $N = 4$, $d = 1$ supermultiplet which also admits a natural description in harmonic superspace. The off-shell content of this supermultiplet is $(\mathbf{4}, \mathbf{4}, \mathbf{0})$.

The $N = 4$, $d = 1$ harmonic superspace $\mathbf{HR}^{1+2|4}$ contains in its bosonic sector two extra harmonic coordinates representing a sphere $S^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$ where SU(2) is one of two commuting SU(2) factors comprising the full R-symmetry group of $N = 4$, $d = 1$ Poincaré supersymmetry [14].[1] It was introduced in [28] (see also [29]) in order to construct an $N = 4$ superextension of the KdV hierarchy, but was never utilized for $d = 1$ sigma model building.

The paper is organized as follows. In sections 2 and 3 we recollect the necessary facts about the standard $N = 4$, $d = 1$ superspace and its SU(2) harmonic extension. Then in sections 4 and 5 we present the harmonic superspace formulation of $N = 4$ SQM models associated with the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ supermultiplet, with the emphasis on the superconformally invariant model considered in [19]. We show that the kinetic and potential terms of its

[1] In principle, one can 'harmonize' both SU(2) factors and consider a bi-harmonic superspace of the type employed e.g. in [26] and [27]. We limit our study here to the simplest case of just one set of SU(2) harmonic variables.

superfield action admit a transparent presentation in harmonic superspace. In particular, the $N = 4$ superconformally invariant potential term can be written as an integral over the (1|2)-dimensional analytic subspace $\mathbf{AR}^{1+2|2} \subset \mathbf{HR}^{1+2|4}$, and it is the simplest $d = 1$ analog of the $d = 4$ harmonic superspace action for the *improved* $N = 2$ *tensor multiplet* [30, 31]. We also present the general off-shell form of an $N = 4$ supersymmetric (but generically not conformally invariant) superpotential for this type of $N = 4$ SQM models. It is also given by an integral over the analytic superspace which thus provides the unique possibility to write the superpotential in a manifestly $N = 4$ supersymmetric way. The harmonic superspace representation allows one to easily recognize the general constraints which $N = 4$ supersymmetry imposes on the purely potential term in the component action and on the related term which describes the coupling to an external three-dimensional gauge potential. In section 6 we discuss another variant of $N = 4$ SQM associated with a $d = 1$ analog of the four-dimensional $N = 2$ *hypermultiplet*. It carries four physical bosons and four physical fermions off-shell and has no auxiliary fields at all.[2] It also admits a simple description as the analytic harmonic $N = 4$, $d = 1$ superfield. We construct superconformally invariant actions as well as general actions for this multiplet and find the general restrictions on the relevant target-space metric and background one-form potential. We also discover an unexpected relation of this $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet with the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet.

2. Preliminaries: the standard $N = 4$, $d = 1$ superspace

For further reference, following ref. [19], we quote here some basic relations of the description of the $N = 4$ models of refs. [22, 23, 13, 25, 11, 19] in ordinary $N = 4$ superspace

$$\mathbf{R}^{1|4} = \{t, \theta_i, \bar\theta^k\} \equiv \{z\}\,. \tag{2.1}$$

The standard $N = 4$, $d = 1$ Poincaré supersymmetry and special conformal supersymmetry from the most general $N = 4$, $d = 1$ superconformal group $D(2, 1; \alpha)$ are realized on these coordinates by the following transformations,

$$\delta t = i\left(\theta \cdot \bar\varepsilon - \varepsilon \cdot \bar\theta\right), \qquad \delta\theta_i = \varepsilon_i\,, \qquad \delta\bar\theta^i = \bar\varepsilon^i\,; \tag{2.2}$$

$$\begin{aligned} \delta' t &= -it\left(\epsilon \cdot \bar\theta + \bar\epsilon \cdot \theta\right) + (1 + 2\alpha)\theta \cdot \bar\theta\left(\epsilon \cdot \bar\theta - \bar\epsilon \cdot \theta\right), \\ \delta'\theta_i &= \epsilon_i t - 2i\alpha\theta_i(\theta \cdot \bar\epsilon) + 2i(1 + \alpha)\theta_i(\bar\theta \cdot \epsilon) - i(1 + 2\alpha)\epsilon_i(\theta \cdot \bar\theta)\,, \end{aligned} \tag{2.3}$$

where ε_i and ϵ_i are the corresponding SU(2) doublet transformation parameters.[3] All other transformations in $D(2, 1; \alpha)$ can be obtained by commuting these basic ones (together with their complex conjugates). So the generic $N = 4$ supersymmetric models should respect invariance under (2.2), while the superconformal models in addition should be invariant under (2.3) (perhaps only for some special values of the parameter α). The conformal supergroup $D(2, 1; \alpha)$ includes as a subgroup not only the $N = 4$ Poincaré supergroup but

[2]Beyond harmonic superspace, such an off-shell $N = 4$, $d = 1$ multiplet was discussed e.g. in [10, 32, 33, 16, 8].

[3]We use the short-hand notation $\bar\psi \cdot \xi = \bar\psi^i\xi_i = -\xi_i\bar\psi^i = -\xi \cdot \bar\psi$, $\psi \cdot \xi = \psi^i\xi_i$, $\bar\psi \cdot \bar\xi = \bar\psi_i\bar\xi^i$. The SU(2) indices are raised and lowered with the help of the skew-symmetric tensors $\epsilon_{ik}, \epsilon^{ik}$ ($\epsilon_{12} = -\epsilon^{12} = 1$).

also its R-symmetry group, i.e. SU(2) $\times$ SU(2). Both these SU(2) factors appear in the commutator of two supersymmetries (2.2), (2.3), but in our notation only one SU(2) is manifest, namely the one rotating doublet indices i, j. The other SU(2) mixes θ_i and $\bar\theta_i$. Both SU(2) can be made manifest by passing to the quartet notation $(\theta_i, \bar\theta_i) \equiv \theta^{ia}$ [14], with the second SU(2) then acting on the additional doublet index a. Here we shall not use this notation.

The semi-covariant (fully covariant only under Poincaré supersymmetry) spinor derivatives are defined by

$$D^i = \frac{\partial}{\partial\theta_i} + i\bar\theta^i\partial_t\,, \qquad \overline{D}_i = \frac{\partial}{\partial\bar\theta^i} + i\theta_i\partial_t\,, \qquad \left\{D^i, \overline{D}_j\right\} = 2i\delta^i_j\partial_t\,. \tag{2.4}$$

They properly transform through each other under (2.3) (see [19]).

The measure of integration over $\mathbb{R}^{1|4}$ is defined as

$$\mu \equiv dt d^4\theta = \frac{1}{4}\, dt\,(D)^2(\bar D)^2\,, \qquad \int d^4\theta\,(\theta)^2(\bar\theta)^2 = 4\,, \tag{2.5}$$

where $(D)^2 = D^i D_i$, $(\bar D)^2 = \bar D_i \bar D^i$, and its specific normalization is chosen for further convenience. It is invariant under (2.2) and transforms with a weight 1 (in mass units) under the superconformal transformations (2.3)

$$\delta'\; dt d^4\theta = 2i(\epsilon\cdot\bar\theta + \bar\epsilon\cdot\theta)\, dt d^4\theta\,. \tag{2.6}$$

The basic $N = 4$, $d = 1$ superfield in the version of $N = 4$ mechanics [13, 25, 19] we are considering here is the isovector superfield

$$V^{ik}(z) = V^{ki}(z) \qquad \overline{V^{ik}} = \epsilon_{ii'}\epsilon_{kk'}V^{i'k'}\,, \tag{2.7}$$

which is subject to the constraints

$$D^{(i}V^{kl)} = 0 \quad \bar D^{(i}V^{kl)} = 0\,. \tag{2.8}$$

They leave in V^{ik} the off-shell irreducible component field content $(\mathbf{3}, \mathbf{4}, \mathbf{1})$: a real triplet of physical bosonic fields, a complex doublet of physical fermionic fields, and a singlet auxiliary field. These constraints are consistent with the action of $D(2,1;\alpha)$ supersymmetry provided V^{ik} is transformed as

$$\delta' V^{ij} = -2i\alpha\left[(\epsilon\cdot\bar\theta + \bar\epsilon\cdot\theta)V^{ij} + (\epsilon^{(i}\bar\theta_k - \bar\epsilon_k\theta^{(i})V^{j)k} + (\epsilon_k\bar\theta^{(i} - \bar\epsilon^{(i}\theta_k)V^{j)k}\right]. \tag{2.9}$$

As a consequence, the object $V^2 \equiv V^{ik}V_{ik}$ is transformed as a density of the weight -2α:

$$\delta' V^2 = -4i\alpha\,(\epsilon\cdot\bar\theta + \bar\epsilon\cdot\theta)\, V^2\,. \tag{2.10}$$

Some useful corollaries of the constraints (2.8) are the following,

$$\begin{aligned} (D)^2V^{ik} &= (\bar D)^2V^{ik} = [D,\bar D]V^{ik} = 0\,, \\ (D)^2\left(V^2\right)^{-1/2} &= (\overline{D})^2\left(V^2\right)^{-1/2} = [D,\overline{D}]\left(V^2\right)^{-1/2} = 0\,, \end{aligned} \tag{2.11}$$

$$D_i\,\frac{V^{ij}}{(V^2)^{3/2}} = \overline{D}_i\,\frac{V^{ij}}{(V^2)^{3/2}} = 0\,. \tag{2.12}$$

The general sigma-model type action of V^{ik} possesses only $N=4$, $d=1$ super Poincaré invariance and is given by an integral over $\mathbb{R}^{1|4}$,

$$S_{(V)} = -\gamma \int dt d^4\theta \, L(V)\,, \tag{2.13}$$

where γ is a positive normalization constant and $L(V)$ is an arbitrary function of V^{ik}. In what follows we shall need the bosonic component part of this action. It can be directly obtained using (2.5), (2.8) as

$$\hat{S}_{(V)} = \gamma \int dt H(v) \left(\dot{v}^{ik}\dot{v}_{ik} + \frac{1}{2}F^2 \right)\,, \qquad H(v) = \Delta L(v)\,, \tag{2.14}$$

where

$$\Delta = \frac{\partial^2}{\partial v^{ik}\partial v_{ik}}\,, \qquad v^{ik}(t) = V^{ik}(z)|\,, \qquad F(t) = \frac{i}{3}\bar{D}_i D_k V^{ik}(z)| \tag{2.15}$$

and $|$ denotes the restriction to the point $\theta = \bar{\theta} = 0$. We see that for bosonic physical fields there arises a sigma model on a 3-dimensional conformally flat manifold with the Weyl factor $H(v) = \Delta L(v)$. In the case of n superfields V^{ik}_A, $A = 1, \ldots n$, and $L(V_1, V_2, \ldots)$ in (2.13) one obtains the following generalization of (2.14) [11]:

$$\hat{S}^{\,n}_{(V)} = \gamma \int dt \left\{ H^{AB}(v) \left(\dot{v}^{ik}_A \dot{v}_{B\,ik} + \frac{1}{2}F_A F_B \right) + 2G^{[AB]}_{(ik)} \left(\dot{v}^{i}_{A\,j}\dot{v}^{jk}_B + F_{[A}\dot{v}^{ik}_{B]} \right) \right\} \tag{2.16}$$

$$H^{AB} = \Delta^{AB} L(v_1, v_2, \ldots)\,, \qquad \Delta^{AB} = \frac{\partial^2}{\partial v^{ik}_A \partial v_{B\,ik}}\,, \qquad G^{[AB]}_{(ik)} = \epsilon^{mn} \frac{\partial^2 L}{\partial v^{m(i}_A \partial v^{k)n}_B}\,. \tag{2.17}$$

The eventual $3n$-dimensional target space metric arises after integrating out the auxiliary field F_A. Its explicit form is not too illuminating. We only point out that in the sigma model target space we encounter a special type of $3n$-dimensional geometry which is a generalization of the conformally-flat 3-geometry in the sense that both are fully specified by a scalar real function $L(v)$ or $L(v_1, v_2, \ldots)$. In some detail this geometry was studied in [11].[4] Note that for the special case of $L(V_1, V_2, \ldots) = \sum_A L(\tilde{V}_A)$ where $\tilde{V}^{ik}_A$ are some linear combinations of the original V^{ik}_A, the second term in (2.16) disappears and the target space metric is drastically simplified. Such systems were considered e.g. in [34, 5].

As was shown in [19], for one V^{ik} the $N=4$ superconformally invariant models in superspace correspond to the following specific choice of the function $L(V)$ in (2.13),

$$S^{\rm conf}_{(V)}(\alpha) = -\gamma \int dt d^4\theta \, (V^2)^{1/2\alpha} \qquad \text{for} \quad \alpha \neq -1\,, \tag{2.18}$$

$$S^{\rm conf}_{(V)}(\alpha) = \frac{1}{2}\gamma \int dt d^4\theta \, (V^2)^{-1/2} \ln(V^2) \qquad \text{for} \quad \alpha = -1\,. \tag{2.19}$$

The second invariant corresponds to the special case when $D(2,1;\alpha)$ becomes isomorphic to a semi-direct product of SU$(1,1|2)$ and second R-symmetry SU(2) group. The first invariant exists in this case too, but in virtue of the relations (2.11) it is identically vanishing. Both invariants yield similar bosonic lagrangians calculated by the general formula (2.14) (with

[4] Actually, it is a geometry with torsion which appears in fermionic terms [11] and it is a generalization of the so-called weak HKT geometry [32].

the auxiliary fields eliminated)

$$\hat{S}^{\rm conf}_{(V)}(\alpha) = \gamma\,\frac{(1+\alpha)}{\alpha^2}\int dt (v^2)^{\frac{1-2\alpha}{2\alpha}}\dot{v}^{ik}\dot{v}_{ik}\,, \qquad \hat{S}^{\rm conf}_{(V)}(\alpha=-1) = \gamma\int dt (v^2)^{-3/2}\dot{v}^{ik}\dot{v}_{ik}\,. \tag{2.20}$$

In [19] we have also shown the existence of a non-trivial $N=4$ superconformally invariant superfield potential term which in the bosonic sector yields a combination of two well known $d=1$ conformal invariants: the standard potential of conformal mechanics [35] and the coupling of a non-relativistic particle in $\mathbb{R}^3$ to the vector potential of a Dirac magnetic monopole [36]. This superinvariant was presented in two equivalent forms: in $N=2$, $d=1$ superspace (where its $N=4$ supersymmetry is non-manifest) and in the full $N=4$ superspace (in terms of an unconstrained prepotential solving (2.8)). We shall demonstrate that this superpotential admits a nice manifestly $N=4$ supersymmetric representation in the analytic subspace of the $d=1$ harmonic superspace to be defined below. We shall see that this object is a representative of the whole class of superpotentials which in general respect only $N=4$, $d=1$ Poincaré supersymmetry and naturally 'live' in the analytic harmonic superspace.

3. $N=4$, $d=1$ harmonic superspace

The $N=4$, $d=1$ harmonic superspace $\mathbf{HR}^{1+2|4}$ [28] is obtained by adding to the $N=4$, $d=1$ superspace coordinates (2.1) a set of harmonic coordinates $u^\pm_i$ parametrizing the sphere $S^2 \sim \mathrm{SU}(2)/\mathrm{U}(1)$, with SU(2) being the R-symmetry group which acts on the doublet indices i,j:

$$u^+_i, u^-_k \in \mathrm{SU}(2)\,, \qquad u^{+i}u^-_i = 1\,, \qquad u^+_i u^-_k - u^+_k u^-_i = \epsilon_{ik}\,. \tag{3.1}$$

Then one can choose the so called analytic basis in $\mathbf{HR}^{1+2|4} = \mathbb{R}^{1|4}\times S^2$

$$\mathbf{HR}^{1+2|4} = \{t_A, \theta^+, \bar\theta^+, \theta^-, \bar\theta^-, u^+_i, u^-_k\} \equiv \{z_A, u^+_i, u^-_k\} \equiv \{\zeta, u^+_i, u^-_k, \theta^-, \bar\theta^-\}\,, \tag{3.2}$$

where

$$\theta^\pm = \theta^i u^\pm_i\,, \qquad \bar\theta^\pm = \bar\theta^i u^\pm_i \qquad \text{and} \qquad t_A = t - i(\theta^+\bar\theta^- + \theta^-\bar\theta^+)\,. \tag{3.3}$$

The original basis will be referred to as the central basis. The analytic basis makes manifest the existence of an important subspace in $\mathbf{HR}^{1+2|4}$, the analytic superspace $\mathbf{AR}^{1+2|2}$ which is a quotient of (3.2) by $\{\theta^-, \bar\theta^-\}$, i.e.

$$\mathbf{AR}^{1+2|2} = \{\zeta, u\} = \{t_A, \theta^+, \bar\theta^+, u^+_i, u^-_k\}\,. \tag{3.4}$$

Its basic feature is that it is closed under the action of $N=4$, $d=1$ Poincaré supersymmetry (2.2) (and under the $D(2,1;\alpha)$ transformations, see below),

$$\delta\theta^+ = \varepsilon^i u^+_i \equiv \varepsilon^+\,, \qquad \delta\bar\theta^+ = \bar\varepsilon^i u^+_i \equiv \bar\varepsilon^+\,, \qquad \delta t_A = -2i\left(\varepsilon^-\bar\theta^+ + \theta^+\bar\varepsilon^-\right), \qquad \delta u^\pm_i = 0\,. \tag{3.5}$$

This property is closely related to the fact that the harmonic projections of the spinor covariant derivatives (2.4),

$$D^{\pm} \equiv D^i u_i^{\pm}\,, \qquad \bar{D}^{\pm} \equiv \bar{D}^i u_i^{\pm}\,, \tag{3.6}$$

take the following explicit form in the analytic basis:

$$D^{+} = \frac{\partial}{\partial\theta^{-}}\,, \qquad \bar{D}^{+} = -\frac{\partial}{\partial\bar{\theta}^{-}}\,, \qquad D^{-} = -\frac{\partial}{\partial\theta^{+}} + 2i\bar{\theta}^{-}\partial_A\,, \qquad \bar{D}^{-} = \frac{\partial}{\partial\bar{\theta}^{+}} + 2i\theta^{-}\partial_A\,. \tag{3.7}$$

We see that $D^{+}, \bar{D}^{+}$ become partial derivatives. Then, the covariant irreducibility conditions for some superfield Φ given on $\mathbf{HR}^{1+2|4}$,

$$D^{+}\Phi(z,u) = \bar{D}^{+}\Phi(z,u) = 0\,, \tag{3.8}$$

are recognized in the analytic basis as Grassmann analyticity conditions. The latter state that in this basis $\Phi(z,u)$ is independent of $\theta^{-}, \bar{\theta}^{-}$:

$$\frac{\partial}{\partial\theta^{-}}\Phi(z_A,u) = \frac{\partial}{\partial\bar{\theta}^{-}}\Phi(z_A,u) = 0 \qquad \Longrightarrow \qquad \Phi(z_A,u) = \phi(\zeta)\,. \tag{3.9}$$

In general, analytic superfields can carry external charges with respect to the harmonic $\mathrm{U}(1)$ (denominator of $\mathrm{SU}(2)/\mathrm{U}(1) \sim S^2$). These superfields are assumed to admit harmonic expansions on S^2, running over integer isospins for even external $\mathrm{U}(1)$ charges and over half-integer ones for odd charges.

An important property of both the harmonic superspace and its analytic subspace is their reality under the generalized involution which is the product of ordinary complex conjugation and Weyl reflection of S^2 (antipodal transformation). Details can be found in [21]; here we only give the transformations of the coordinates and spinor derivatives in the analytic basis,

$$\widetilde{t_A} = t_A\,, \qquad \widetilde{\theta^{\pm}} = \bar{\theta}^{\pm}\,, \qquad \widetilde{\bar{\theta}^{\pm}} = -\theta^{\pm}\,, \qquad \widetilde{(u_i^{\pm})} = u^{\pm i}\,, \qquad \widetilde{(u^{\pm i})} = -u_i^{\pm}\,, \tag{3.10}$$

$$\widetilde{D^{\pm}} = -\bar{D}^{\pm}\,, \qquad \widetilde{\bar{D}^{\pm}} = D^{\pm}\,. \tag{3.11}$$

Using this involution, one can impose reality conditions on the analytic superfields. The involution squares to 1 on the objects with even $\mathrm{U}(1)$ charges and to -1 on those with odd charges.

Another important ingredient of the harmonic formalism are the covariant derivatives on the harmonic S^2. In the central basis they are

$$D^{\pm\pm} = u_i^{\pm}\frac{\partial}{\partial u_i^{\mp}} \equiv \partial^{\pm}\,, \qquad D^{0} = u_i^{+}\frac{\partial}{\partial u_i^{+}} - u_i^{-}\frac{\partial}{\partial u_i^{-}} \equiv \partial^{0}\,, \tag{3.12}$$

$$[D^{++}, D^{--}] = D^{0}\,, \qquad [D^{0}, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,. \tag{3.13}$$

The same objects in the analytic basis read

$$D^{++} = \partial^{++} - 2i\theta^{+}\bar{\theta}^{+}\partial_A + \theta^{+}\frac{\partial}{\partial\theta^{-}} + \bar{\theta}^{+}\frac{\partial}{\partial\bar{\theta}^{-}}\,, \tag{3.14}$$

$$D^{--} = \partial^{--} - 2i\theta^{-}\bar{\theta}^{-}\partial_A + \theta^{-}\frac{\partial}{\partial\theta^{+}} + \bar{\theta}^{-}\frac{\partial}{\partial\bar{\theta}^{+}}\,, \tag{3.15}$$

$$D^{0} = \partial^{0} + \left(\theta^{+}\frac{\partial}{\partial\theta^{+}} + \bar{\theta}^{+}\frac{\partial}{\partial\bar{\theta}^{+}}\right) - \left(\theta^{-}\frac{\partial}{\partial\theta^{-}} + \bar{\theta}^{-}\frac{\partial}{\partial\bar{\theta}^{-}}\right). \tag{3.16}$$

These operators are invariant under the $\widetilde{\ }$ conjugation. As is seen from its explicit form, the covariant derivative D^{++} commutes with the spinor derivatives $D^+, \bar{D}^+$ in (3.8) and so preserves the Grassmann harmonic analyticity: acting on an analytic superfield, it produces an analytic superfield. In contrast, the derivative D^{--} does not preserve the analyticity. The operator D^0 counts the external U(1) charges of the harmonic superfields. Some important (anti)commutation relations to be used below are

$$[D^{\pm\pm}, D^{\mp}] = D^{\pm}\,, \qquad [D^{\pm\pm}, \bar{D}^{\mp}] = \bar{D}^{\pm} \qquad \text{and} \qquad \{D^+, \bar{D}^-\} = -\{D^-, \bar{D}^+\} = 2i\,\partial_t\,. \tag{3.17}$$

Now let us see how the superconformal $D(2,1;\alpha)$ symmetry acts in the harmonic superspace. We start with the ansatz

$$\delta' u_i^+ = \Lambda^{++} u_i^- \qquad \text{and} \qquad \delta' u_i^- = 0\,, \tag{3.18}$$

which is consistent with the defining condition $u^{+i}u_i^- = 1$ and is the typical superconformal transformation law of harmonic variables [21]. Though this is not consistent with ordinary complex conjugation, it nicely matches with the above-mentioned generalized conjugation $\widetilde{\ }$ which substitutes the ordinary one in harmonic superspace. Then we require the analytic subspace (3.4) to be closed under the transformations (2.3) and (3.18) with taking into account the relations (3.3). This requirement proves to uniquely fix the transformations of $\theta^+, \bar{\theta}^+$ and the function Λ^{++} as

$$\begin{aligned} \delta'\theta^+ &= \epsilon^+ t_A + 2i(1{+}\alpha)\epsilon^-(\theta^+\bar{\theta}^+)\,, \qquad \delta'\bar{\theta}^+ = \bar{\epsilon}^+ t_A + 2i(1{+}\alpha)\bar{\epsilon}^-(\theta^+\bar{\theta}^+)\,, \qquad &(3.19)\\ \Lambda^{++} &= -2i\alpha(\epsilon^+\bar{\theta}^+ {-} \bar{\epsilon}^+\theta^+) \equiv D^{++}\Lambda\,, \qquad \Lambda = -2i\alpha(\epsilon^-\bar{\theta}^+ {-} \bar{\epsilon}^-\theta^+)\,. &(3.20) \end{aligned}$$

so that $(D^{++})^2\Lambda = 0$. It is easy to find the transformation laws of $\theta^-, \bar{\theta}^-$ and t_A:

$$\begin{aligned} \delta'\theta^- &= \epsilon^- t_A + 2i[\,(1{+}\alpha)\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+\,] + 2i\alpha\,\bar{\epsilon}^-\theta^-\theta^+ - 2i(1{+}\alpha)\epsilon^+\theta^-\bar{\theta}^-\,,\\ \delta' t_A &= -2i t_A\,(\epsilon^-\bar{\theta}^+ {-} \bar{\epsilon}^-\theta^+)\,, \qquad \delta'\bar{\theta}^- = \widetilde{\delta'\theta^-}\,. \end{aligned} \tag{3.21}$$

We see that $\mathbf{AR}^{1+2|2}$ defined in (3.4) is closed under $N=4$ superconformal transformations at any value of α, in contrast to the standard left-chiral subspace of $\mathbb{R}^{1|4}$, i.e. $\mathbf{C}^{1|2} = \{t_c, \theta_i\}$, which is closed only for $\alpha = -1$, i.e. with respect to SU(1,1|2) supersymmetry [19]. Thus we can define $N=4$, $d=1$ superconformally-covariant *analytic* harmonic superfields for any value of α, while the analogous *chiral* superfields can be consistently defined only for the choice of SU(1,1|2) as the $N=4$, $d=1$ superconformal group. It is worth noting that the Grassmann coordinate transformation laws (3.19), (3.21) are drastically simplified at $\alpha = -1$.

Now it is easy to find the $D(2,1;\alpha)$ transformation rules of the different covariant derivatives and to check e.g. that $D^+, \bar{D}^+$ in the analytic basis transform through each other, thereby preserving the Grassmann analyticity (3.8). For our further discussion we shall need the transformation rules of harmonic derivatives D^{++}, D^{--} defined in (3.14), (3.15),

$$\delta' D^{++} = -\Lambda^{++}\,D^0 \qquad \text{and} \qquad \delta' D^{--} = -(D^{--}\Lambda^{++})D^{--}\,. \tag{3.22}$$

The U(1) charge counter D^0 is invariant, which can be proved e.g. by varying the relations of the D-algebra (3.13). The whole set of $D(2,1;\alpha)$ transformations can be obtained by repeatedly commuting the above δ'-transformations with each other and with those of $N=4$, $d=1$ Poincaré supersymmetry. For any element of $D(2,1;\alpha)$, the transformation of the harmonics and those of $D^{\pm\pm}$ have the same form, with all transformation parameters being properly accommodated by the superfunction $\Lambda^{++}(\zeta,u)$. The latter satisfies the same differential constraints as in the particular case (3.20), namely

$$\Lambda^{++}=D^{++}\Lambda \qquad \text{and} \qquad D^{++}\Lambda^{++}=0\,. \tag{3.23}$$

The measures of integration over the full harmonic superspace and over its analytic subspace, denoted by μ_H and μ_A, are defined as

$$\begin{aligned} \mu_H &= du\,\mu = dudtd^4\theta = dudt_A(D^-\bar D^-)(D^+\bar D^+) = \mu_A^{--}(D^+\bar D^+)\,, \\ \mu_A^{--} &= dud\zeta^{--} = dudt_A D^-\bar D^-\,. \end{aligned} \tag{3.24}$$

They are evidently invariant under the $N=4$ Poincaré supersymmetry and have the following transformation properties under the superconformal transformations:

$$\delta'\mu_A^{--} = \left(\partial_A\delta' t_A + \partial^{--}\Lambda^{++} - \partial_{\theta^+}\delta'\theta^+ - \partial_{\bar\theta^+}\delta'\bar\theta^+\right)\mu_A^{--} = 0\,, \tag{3.25}$$

$$\begin{aligned} \delta'\mu_H &= \left(\partial_A\delta' t_A + \partial^{--}\Lambda^{++} - \partial_{\theta^+}\delta'\theta^+ - \partial_{\bar\theta^+}\delta'\bar\theta^+ - \partial_{\theta^-}\delta'\theta^- - \partial_{\bar\theta^-}\delta'\bar\theta^-\right)\mu_H \\ &= 2i\left[(1-\alpha)(\epsilon^-\bar\theta^+ - \bar\epsilon^-\theta^+) - (1+\alpha)(\epsilon^+\bar\theta^- - \bar\epsilon^+\theta^-)\right]\mu_H\,. \end{aligned} \tag{3.26}$$

Recall that the integration over the harmonic $S^2=\{u_i^+,u_k^-\}$ is normalized so that

$$\int du\,1 = 1\,, \tag{3.27}$$

and the integral of any other irreducible monomial of the harmonics is vanishing [21].

Finally, let us point out that the $N=4$, $d=1$ harmonic superspace is by no means a dimensional reduction of the $N=2$, $d=4$ one. Indeed, $N=2$, $d=4$ supersymmetry amounts to $N=8$ supersymmetry in $d=1$, so a $d=1$ reduction of theories in $N=2$, $d=4$ harmonic superspace would yield $N=8$ supersymmetric SQM models. On the other hand, the $N=4$, $d=1$ supersymmetry can be regarded as a reduction of the $N=1$, $d=4$ one. No standard harmonic superspaces can be defined for the latter because of lacking of non-abelian R-symmetry group in the $N=1$, $d=4$ case. However, such symmetry $(\mathrm{SU}(2)\times\mathrm{SU}(2))$ appears after reduction to $d=1$ and this makes it possible to define harmonic superspace for $N=4$, $d=1$ supersymmetry.

4. Constrained analytic $N=4$, $d=1$ superfields

Generically, unconstrained analytic $N=4$, $d=1$ superfields contain an infinite number of standard $d=1$ fields with growing isospin, due to the harmonic S^2 expansions of the u-dependent component fields in the $\theta^+,\bar\theta^+$ expansion. While in higher dimensions the presence of these infinite tails of auxiliary fields sometimes turns out to be crucial for the

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existence of off-shell superfield actions,[5] for the time being it is far from obvious which role such unconstrained superfields could play in $N = 4$, $d = 1$ SQM models. On the other hand, one can suitably constrain the harmonic dependence by imposing on these superfields harmonic constraints which make use of the analyticity-preserving harmonic derivative D^{++}. In this way not only some known $N = 4$, $d = 1$ multiplets can be recovered as constrained $N = 4$, $d = 1$ harmonic analytic superfields, but also new types of such multiplets can be exhibited. To be aware of all such multiplets is important e.g. for $N > 4$, $d = 1$ model building.

Let us start by showing that the constraints (2.8) are nothing but the Grassmann harmonic analyticity conditions. One introduces a harmonic superfield $V^{++}(z, u)$ and subjects it to the following set of constraints in harmonic superspace:

$$(a) \quad D^+V^{++} = \bar{D}^+V^{++} = 0\,; \qquad (b) \quad D^{++}V^{++} = 0\,. \tag{4.1}$$

The constraint (4.1b) in the central basis simply implies V^{++} to be quadratic in the harmonics:

$$D^{++}V^{++}(z, u) = 0 \qquad \Longrightarrow \qquad V^{++}(z, u) = V^{ik}(z)u^+_i u^+_k\,. \tag{4.2}$$

Then (4.1a), after stripping off the harmonics, yields just (2.8). The virtue of this equivalent form (4.1) for the basic constraints (2.8) is revealed after passing to the analytic basis (3.2). In it, eqs. (4.1a) are just the Grassmann analyticity conditions

$$V^{++} = V^{++}(\zeta, u) \tag{4.3}$$

while (4.1b) cuts the infinite harmonic expansion of $V^{++}(\zeta, u)$ to finite size, leaving in it just the irreducible $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ field content

$$V^{++}(\zeta, u) = v^{ik}(t_A)u^+_i u^+_k + \theta^+\psi^i(t_A)u^+_i + \bar{\theta}^+\bar{\psi}^i(t_A)u^+_i + i\theta^+\bar{\theta}^+\left(F(t_A) + 2\dot{v}^{ik}(t_A)u^+_i u^-_k\right), \tag{4.4}$$

where v^{ik} and F are the physical and auxiliary bosonic fields, respectively. In order to unclutter the notation, we shall sometimes omit the index A on the analytic-basis time variable, hoping that this will not give rise to confusion. The difference between t_A and t should be taken into account when rewriting the analytic superfields in the central basis.

Using the transformation laws (2.9) and (3.18) with keeping in mind the definitions (3.20) it is straightforward to find that $V^{++}(\zeta, u)$ transforms under $D(2, 1; \alpha)$ like

$$\delta' V^{++} = 2\Lambda\, V^{++}\,. \tag{4.5}$$

Using the properties (3.22), (3.23) it is also easy to check the covariance of the harmonic constraint (4.1b) with respect to these transformations. For further use we shall define non-analytic harmonic superfields V^{--} and V^{+-} by

$$V^{+-} = \frac{1}{2}D^{--}V^{++}\,, \qquad V^{--} = \frac{1}{2}(D^{--})^2V^{++} \tag{4.6}$$

[5] Take, for instance, the off-shell actions of the hypermultiplets in $N = 2$, $d = 4$ supersymmetry [20, 21].

the transformation properties of which can be easily found using (4.5) and (3.22):

$$\begin{aligned}\delta' V^{+-} &= (2\Lambda - D^{--}\Lambda^{++})\, V^{+-} + (D^{--}\Lambda) V^{++}\,, \\ \delta' V^{--} &= 2(\Lambda - D^{--}\Lambda^{++})\, V^{--} + 2D^{--}\Lambda V^{+-}\,. \end{aligned} \tag{4.7}$$

It is worth noting that

$$V^{++}V^{--} - (V^{+-})^2 = \frac{1}{2} V^{ik} V_{ik}\,, \tag{4.8}$$

$$\delta'[V^{++}V^{--} - (V^{+-})^2] = (2\Lambda - D^{--}\Lambda^{++})[V^{++}V^{--} - (V^{+-})^2]\,, \tag{4.9}$$

which may be checked to coincide with the transformation law (2.10).

So much for the harmonic superspace formulation of the $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet V^{++}. In the next section we shall demonstrate how the sigma-model actions (2.13), (2.18), (2.19) can also be rewritten in harmonic superspace. We shall then construct the most general manifestly $N=4$ invariant superpotential term and its superconformal version as integrals over the analytic superspace (3.4). In the remainder of the present section we briefly dwell on some direct generalizations of V^{++}.

One may consider a general analytic superfield $q^{(+n)}$ with harmonic U(1) charge equal to $+n$ and impose on it the same constraint as (4.1b),

$$D^{++} q^{(+n)} = 0\,. \tag{4.10}$$

It is easy to check that, unless $n = 0$, this constraint defines an off-shell $N = 4$, $d = 1$ multiplet with the field content $[\mathbf{2(n{+}1)}, \mathbf{4n}, \mathbf{2(n{-}1)}]$ (for even n one may halve this content by imposing a reality condition). For $n \leq 0$, eq. (4.10) constrains $q^{(+n)}$ to be a constant or to vanish, in accord with one of the basic principles of the harmonic superspace formalism [21]: $D^{++} f^{(-n)} = 0 \;\Rightarrow\; f^{(-n)} = 0$. The superfields $q^{(+n)}$ for any $n > 0$ are superconformal, in the sense that the whole $D(2,1;\alpha)$ at any α admits a self-consistent realization on them. This realization uniquely follows from requiring (4.10) to be covariant with respect to the $D(2,1;\alpha)$ transformation (3.22):

$$\delta' q^{(+n)} = n\,\Lambda\, q^{(+n)} \tag{4.11}$$

(one should take into account here the property $\Lambda^{++} = D^{++}\Lambda$). Note that the first component of Λ is the parameter of dilatations times $\alpha/2$, so the dilatation weight of $q^{(+n)}$ (in the mass units) turns out to be strictly related with its harmonic U(1) charge:

$$d(q^{(+n)}) = -\frac{\alpha}{2}\, n\,. \tag{4.12}$$

In particular, the dilatation weight of $V^{++} {\equiv} q^{(+2)}$ is $-\alpha$.

It is interesting to consider the particular value $n = 1$. In this case we deal with the superfield $q^{+}(\zeta, u)$ having the off-shell field content $(\mathbf{4},\mathbf{4},\mathbf{0})$ — the only possibility for an off-shell representation of $N = 4$, $d = 1$ supersymmetry without any auxiliary fields! It is convenient to group q^{+} and $\widetilde{q^{+}}$ into a doublet of an extra (the so-called Pauli-Gürsey) SU(2), [21])

$$(q^{+}_a) = (q^{+}, -\widetilde{q}^{+}) \qquad \text{so that} \quad \widetilde{q^{+}_a} = q^{+a} = \epsilon^{ab} q^{+}_b \qquad \text{for} \quad a, b = 1, 2\,. \tag{4.13}$$

The explicit solution of (4.10) for this case, i.e. of

$$D^{++} q^{+a} = 0\,, \tag{4.14}$$

reads

$$q^{+a}(\zeta, u) = f^{ia}(t_A) u_i^+ + \theta^+ \chi^a(t_A) + \bar\theta^+ \bar\chi^a(t_A) + 2i\theta^+\bar\theta^+ \dot f^{ia}(t_A) u_i^-\,, \tag{4.15}$$

where

$$\overline{(f^{ia})} = \epsilon_{ab}\epsilon_{ik} f^{kb}\,, \qquad \overline{(\chi^a)} = \bar\chi_a\,, \tag{4.16}$$

as a consequence of the reality property (4.13). This $N = 4$, $d = 1$ multiplet already appeared in [7, 32] in component and in $N = 1$ superfield form and also in [16, 8] as a constrained $N = 4$ superfield in the standard $N = 4$, $d = 1$ superspace. Its relation with some other off-shell $N = 4$, $d = 1$ multiplets also having 4 bosonic and 4 fermionic components, in particular with the above $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ one, was discussed at the algebraic level in [33]. We see here that this field has a very simple description as a constrained superfield in analytic harmonic $N = 4$, $d = 1$ superspace. For later use we spell out its $D(2, 1; \alpha)$ transformation law,

$$\delta' q^{+a} = \Lambda\, q^{+a}\,. \tag{4.17}$$

Also note that in the central basis eq. (4.14) and the Grassmann analyticity conditions imply the standard $\mathbb{R}^{1|4}$ constraints

$$q^{+a}(z, u) = q^{ia}(z) u_i^+ \qquad \text{and} \qquad D^{(i} q^{k)a} = \bar D^{(i} q^{k)a} = 0\,. \tag{4.18}$$

They can easily be shown to be equivalent to those employed in [16, 8].

Both by its field content and by its superconformal transformation properties the superfield q^{+a} resembles the analytic hypermultiplet superfield of $N = 2$, $d = 4$ Poincaré supersymmetry. The crucial difference between the two is that in the hypermultiplet case the constraint (4.14) puts this superfield on shell, while in our case it defines an off-shell multiplet without any further dynamical constraints. Another difference is that the transformation (4.17) is a realization of the supergroup $D(2, 1; \alpha)$ which cannot be obtained as a reduction of the $N = 2$, $d = 4$ superconformal group $\mathrm{SU}(2, 2|2)$. Both these crucial distinctions are of course specific features of $N = 4$, $d = 1$ supersymmetry.

In what follows we shall focus on the multiplets V^{++} and q^{+a}, leaaving for the future the study of possible implications of the supermultiplets $q^{(+n)}$ with $n > 2$ in $N = 4$ SQM models. For completeness, we mention that two further $N = 4$, $d = 1$ superfields have been used in constructing $N = 4$ SQM models, those comprising a $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet [12, 15, 14] and a $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplet [37, 12, 14, 38, 5]. The first superfield is real and obeys constraints which are bilinear in spinor derivatives and so cannot be interpreted as Grassmann analyticity conditions, while the second one is simply the chiral $N = 4$, $d = 1$ superfield. Clearly, the natural superspaces to deal with these multiplets are, respectively, the standard and chiral $N = 4$, $d = 1$ superspaces, but not the harmonic superspace.

5. Harmonic superfield actions for V^{++}

5.1 Sigma-model type actions

We begin by discussing the sigma-model actions for V^{++}. When considering the most general sigma-model type action for V^{++}, the harmonic superspace approach does not bring any new features. Indeed, by dimensional arguments, the corresponding action is naturally written as an integral over the whole harmonic superspace,

$$S_{(V)} = -\gamma \int dudtd^4\theta\, \mathcal{L}(V^{++}, V^{+-}, V^{--}, u)\,. \tag{5.1}$$

Then, choosing the central basis, substituting $V^{\pm\pm} = V^{ik} u_i^{\pm} u_k^{\pm}$ and $V^{+-} = V^{ik} u_i^{+} u_k^{-}$ into (5.1) and integrating over harmonics we return to the general action (2.13) of the $\mathbb{R}^{1|4}$ superfield $V^{ik}(z)$ with

$$L(V) = \int du \mathcal{L}(V^{++}, V^{+-}, V^{--}, u)$$

(or its $3n$-dimensional target-space generalization in the case of several V^{ik} multiplets).

One may wonder whether self-consistent sigma-model type actions for V^{++} can be written in the harmonic analytic superspace $\mathbf{AR}^{1+2|2}$ (3.4). To single out such a subclass in the set of general actions (2.13), let us start with the free action corresponding to $L(V) = 1/6\, V^{ik}V_{ik}$ and rewrite it as the following integral over the whole harmonic superspace $\mathbf{HR}^{1+2|4}$:

$$S^{free}_{(V)} = -\frac{1}{6}\int dtd^4\theta\, V^2 = -\frac{1}{4}\int dudt_A d^4\theta\, V^{++}(D^{--})^2 V^{++}\,, \tag{5.2}$$

where we made use of the relations (4.8) and integrated by parts with respect to harmonics. Using the relations (3.24) and (3.17), we can further rewrite (5.2) as an integral over the analytic superspace,[6]

$$S^{free}_{(V)} = i\int du\, d\zeta^{--}\, V^{++}\left(D^{--}\partial_t + \frac{i}{2}D^{-}\bar{D}^{-}\right)V^{++}\,. \tag{5.3}$$

Using (3.17), it is easy to check that the differential operator in (5.3) commutes with D^{+} and $\bar{D}^{+}$ and so preserves the analyticity.

The evident analyticity-preserving nonlinear sigma-model extension of (5.3) for n superfields V^{++}_B , $B = 1, 2, \ldots, n$, is as follows,

$$S^{n\,\sigma}_{(V)} = i\int du\, d\zeta^{--}\, \mathcal{L}^{++\,B}(V^{++}_1, \ldots, u)\left(D^{--}\partial_t + \frac{i}{2}D^{-}\bar{D}^{-}\right)V^{++}_B\,. \tag{5.4}$$

The target geometry prepotential $\mathcal{L}^{++B}(V, u)$ is an arbitrary charge 2 function of V^{++}_A and explicit harmonics. The bosonic sigma-model action is easily computed to be of the general form (2.16), (2.17) where the tensors H^{AB} and $G^{[AB]}_{(ik)}$ are expressed as the following harmonic integrals,

$$H^{AB} = \int du\, \partial^{(A}_{++}\mathcal{L}^{++\,B)}(v^{++}, u)\,, \qquad G^{[AB]}_{(ik)} = \int du\, \partial^{[A}_{++}\mathcal{L}^{++\,B]}(v^{++}, u)\, u^{+}_{(i}u^{-}_{k)}\,, \tag{5.5}$$

[6] Hereafter we omit the index A on the time derivative because $\partial_{t_A} = \partial_t$.

where $\partial^A_{++} = \partial/\partial v^{++}_A$. The corresponding function $L(v)$ is given by

$$L(v) = \frac{1}{2}\int du\, \mathcal{L}^{++B}(V^{++},u)V^{--}_B\,. \tag{5.6}$$

Due to the fact that all geometric objects in this special case are expressed through the single harmonic unconstrained prepotential $\mathcal{L}^{++A}$, there appear some relations between them and also further restrictions. In particular, it is easy to check that H^{AB} and $G^{[AB]}_{(ik)}$ obey the generalized harmonicity condition

$$\Delta^{AB}H^{CD} = \Delta^{AB}G^{[CD]}_{(ik)} = 0\,. \tag{5.7}$$

In the 3-dimensional case, we are facing the conformally flat 3-dimensional metric (2.14), with $H(v)$ being now some harmonic function,[7]

$$\Delta H(v) = 0\,. \tag{5.8}$$

Presently, we do not fully understand this special $3n$-geometry. Perhaps it can be obtained as a reduction of some strong $4n$-dimensional HKT geometry [32]. Recall that it is specified by the requirement of $N=4$, $d=1$ supersymmetry and, in addition, by the assertion that the sigma model action admits a representation in analytic harmonic $N=4$, $d=1$ superspace $\{\zeta, u\}$, with the lagrangian being a local function of the analytic superfields V^{++}_A.

5.2 Analytic superpotentials

The actual virtue of the harmonic superspace emerges in the opportunity to write down the general manifestly $N=4$ supersymmetric superpotential term for V^{++}. In the case of a single V^{++} it is given by the following integral over the analytic superspace,

$$S^{sp}_{(V)} = -i\gamma'\int dud\zeta^{--}\, L^{++}(V^{++},u)\,. \tag{5.9}$$

This superpotential is manifestly $N=4$ supersymmetric since the lagrangian is defined on the analytic superspace which is closed under $N=4$ supersymmetry and the integral is taken over this superspace.[8] Thus L^{++} can be an arbitrary function of its arguments, and there is no need to care about the conditions which the bosonic background should satisfy for $N=4$ supersymmetry to be valid. Actually these conditions can now be *derived* from (5.9), passing there to components and considering the bosonic sector. Substituting the analytic basis form (4.4) for V^{++} into (5.9) and integrating over $\theta^+, \bar\theta^+$, we find the general structure of the component bosonic sector of (5.9),

$$\hat{S}^{sp}_{(V)} = \gamma'\int dt\,\left\{F\,\mathcal{V}(v) + \dot v^{ik}\,\mathcal{A}_{ik}(v)\right\}, \tag{5.10}$$

[7] We hope that the use of the term 'harmonic' in two different meanings is not too confusing.

[8] This should be contrasted with a non-manifestly supersymmetric way of writing the superpotential as an integral over the chiral $N=4$, $d=1$ superspace with a non-holomorphic lagrangian density [16].

where the background scalar 'half-potential' $\mathcal{V}$ and the magnetic one-form potential $\mathcal{A}_{ik}$ are given by the following harmonic integrals,

$$\mathcal{V}(v) = \int du \, \frac{\partial L^{++}}{\partial v^{++}} \,, \qquad \mathcal{A}_{ik}(v) = 2 \int du \, u^+_{(i} u^-_{k)} \frac{\partial L^{++}}{\partial v^{++}} \,, \qquad v^{++} = v^{ik} u^+_i u^+_k \,. \tag{5.11}$$

The term 'half-potential' is related to the fact that the genuine scalar potential W appears as the result of eliminating the auxiliary field $F(t)$ in the sum of the sigma-model action (2.14) and (5.10) as

$$W(v) = -\frac{1}{2} \frac{\gamma'^2}{\gamma} \frac{(\mathcal{V}(v))^2}{H(v)} \,. \tag{5.12}$$

We observe that both objects, $\mathcal{V}(v)$ and $\mathcal{A}_{ik}$, are expressed through the same analytic 'prepotential' $L^{++}(v^{++}, u)$ which is required to have harmonic U(1) charge $+2$ but is unconstrained otherwise. Choosing it at will, we always obtain a 'half-potential' and a background vector potential compatible with $N = 4$, $d = 1$ supersymmetry. Inversely, the representation (5.11) allows one to find the most general constraints which $\mathcal{V}$ and $\mathcal{A}_{ik}$ should obey in order to admit an $N = 4$ supersymmetric extension. These are easily computed to be

$$\partial_{ik} \mathcal{A}_{lt} - \partial_{lt} \mathcal{A}_{ik} = \epsilon_{il} \, \partial_{kt} \mathcal{V} + \epsilon_{kt} \, \partial_{il} \mathcal{V} \qquad \text{and} \qquad \Delta \mathcal{V} = 0 \,. \tag{5.13}$$

This system is recognized as the static-solution (monopole) ansatz for a self-dual Maxwell field in $\mathbb{R}^4$. In fact, the set of equations (5.13) is the same as in the famous Gibbons-Hawking multi-center ansatz for 4-dimensional hyper-Kähler metrics [39, 40]. Leaving aside the issue of boundary condition in the target space, any solution of these equations, up to gauge freedom, give us a 'half-potential' and a background vector potential admissible from the point of view of $N = 4$ supersymmetry. The simplest choice is the 'Fayet-Iliopoulos term' [25]

$$L^{++} = c \, V^{++} \quad \text{with} \quad c = \text{const} \in \mathbb{R} \qquad \Longrightarrow \qquad \mathcal{V} = c \,, \qquad \mathcal{A}_{ik} = 0 \,, \tag{5.14}$$

and the only effect of adding such a superpotential term consists in producing a scalar potential

$$W \sim -\frac{c^2}{H(v)} \tag{5.15}$$

in the case of a non-trivial metric function $H(v)$.

A more complicated and interesting example is the well known multi-center solution

$$\mathcal{V} = c_0 + \sum_A \frac{c_A}{|\mathbf{v} - \mathbf{v}_A|} \,, \tag{5.16}$$

where $\mathbf{v} = (v^{ik})\,, \mathbf{v} \cdot \mathbf{v} = v^{ik} v_{ik}$. The corresponding $\mathcal{A}_{ik}$ can be straightforwardly found, but we do not give them here. The notorious example of this kind is the spherically symmetric solution

$$\mathcal{V} = c_0 + |\mathbf{v}|^{-1} \tag{5.17}$$

for which

$$F_{ik,lt} := \partial_{ik}\mathcal{A}_{lt} - \partial_{lt}\mathcal{A}_{ik} = -(\,\epsilon_{il}v_{kt} + \epsilon_{kt}v_{il}\,)\,|\mathbf{v}|^{-3}\,. \tag{5.18}$$

This is the Dirac magnetic monopole. One can easily find the corresponding prepotential $L^{++}(v^{++},u)$ and compute the relevant vector potential $\mathcal{A}_{ik}$ via eq. (5.11). We shall do this below, after discussing $N=4$ superconformally invariant superpotentials.

Note that the $N=4$ supersymmetry constraints (5.11) were derived for the first time in [22] in an on-shell hamiltonian approach (see also [23]).

5.3 $N=4$ superconformally invariant superpotential

In order to find an $N=4$ superconformally invariant potential for V^{++} we shall follow a strategy which was applied for constructing the $N=2$, $d=4$ analytic harmonic superfield action of the improved tensor multiplet in [30] and the $N=(4,4)$, $d=2$ harmonic superspace action of the $N=(4,4)$, SU(2) WZW model in [26]. Actually, the action we are going to construct is the true $d=1$ analog of the general harmonic analytic action of the $N=2$, $d=4$ improved tensor multiplet. The difference lies, however, in the fact that the latter produces the sigma-model type action for physical bosons, with two derivatives on the latter, while in the case under consideration we end up with the sum of the scalar potential and the coupling to the background vector potential, with only one time derivative on the physical bosonic field v^{ik}.

The trick of [30] adapted to the given case works as follows. Let us split

$$V^{++} = \hat{V}^{++} + c^{++}\,, \tag{5.19}$$

where

$$c^{\pm\pm} := c^{ik}u^{\pm}_i u^{\pm}_k\,, \qquad c^{+-} := c^{ik}u^{+}_i u^{-}_k\,, \qquad c^{++}c^{--} - (c^{+-})^2 = \frac{1}{2}c^{ik}c_{ik} = 1\,, \tag{5.20}$$

and the constant 3-vector c^{ik} satisfies the same reality condition as V^{ik}. The $D(2,1;\alpha)$ transformations (3.18), (4.5) imply for $\hat{V}^{++}$ the inhomogeneous transformation law

$$\delta'\hat{V}^{++} = 2\Lambda(\hat{V}^{++} + c^{++}) - 2\Lambda^{++}c^{+-}\,. \tag{5.21}$$

Then, one searches for $L^{++}_{\rm conf}(V^{++},u)$ as a power series in $X := c^{--}\hat{V}^{++}$:

$$L^{++}_{\rm conf} = \hat{V}^{++}\sum_{n=0}^{\infty} b_n\,X^n =: \sum_{n=0}^{\infty} L^{++}_{(n)}\,. \tag{5.22}$$

The basic idea is to take advantage of the inhomogeneity of the transformation law (5.21) in order to cancel the variations between the adjacent terms in the sum (5.22), integrating by parts with respect to D^{++} and using the relations

$$D^{++}\hat{V}^{++} = 0\,, \qquad D^{++}c^{++} = 0\,, \qquad D^{++}c^{--} = 2c^{+-}\,, \qquad D^{++}c^{+-} = c^{++}\,. \tag{5.23}$$

As the necessary condition for such cancellations, there will appear recurrence relation involving the coefficients in (5.22) which will be used to restore the precise functional form of $L^{++}_{\rm conf}$.

Keeping in mind that the integration measure of analytic superspace is invariant under $D(2,1;\alpha)$ transformations, the variation of the first term in (5.22), up to a total harmonic derivative, reads

$$\delta' L^{++}_{(0)} = 2b_0\,\Lambda\hat{V}^{++}\,. \tag{5.24}$$

Analogously, the variation of the next term can be cast in the form

$$\delta' L^{++}_{(1)} = 4b_1\Lambda c^{++}c^{--}\hat{V}^{++} + 4b_1\Lambda c^{--}(\hat{V}^{++})^2\,. \tag{5.25}$$

Using the relations (5.20) and (5.23), it is easy to find

$$c^{++}c^{--} = \frac{1}{12}(D^{++})^2(c^{--})^2 + \frac{2}{3}\,. \tag{5.26}$$

Substituting this into the first term of (5.25), we find that, up to a total derivative, it reduces to

$$\frac{8}{3}b_1\Lambda\,\hat{V}^{++}\,. \tag{5.27}$$

Comparing this with (5.24), we find the condition of their cancellation to be

$$b_1 = -\frac{3}{4}b_0\,. \tag{5.28}$$

Similarly, the remaining part of $\delta' L^{++}_{(1)}$ is cancelled by the inhomogeneous part of the variation $\delta' L^{++}_{(2)}$, provided that

$$b_2 = -\frac{5}{6}\,b_1\,. \tag{5.29}$$

Continuing this procedure by induction and comparing the variations $\delta' L^{++}_{(n-1)}$ and $\delta' L^{++}_{(n)}$, we find the recurrence relation

$$b_n = -\frac{2n+1}{2n+2}\,b_{n-1}\,. \tag{5.30}$$

Solving the latter, one obtains

$$\begin{aligned} L^{++}_{\text{conf}} &= b_0\,\hat{V}^{++}\sum_{n=0}^{\infty}\frac{(2n+1)!}{(n+1)!\,n!}\left(-\frac{1}{4}X\right)^n \\ &= 2b_0\,\frac{\hat{V}^{++}}{\sqrt{1+c^{--}\hat{V}^{++}}\left(1+\sqrt{1+c^{--}\hat{V}^{++}}\right)}\,. \end{aligned} \tag{5.31}$$

Now it is straightforward to explicitly check that the resulting action

$$S^{\text{sp conf}}_{(V)} = -ib_0\int du d\zeta^{--}\,\frac{2\,\hat{V}^{++}}{\sqrt{1+c^{--}\hat{V}^{++}}\left(1+\sqrt{1+c^{--}\hat{V}^{++}}\right)} \tag{5.32}$$

is invariant under (5.21) up to a total harmonic derivative. This can be done with making use of the formula

$$\delta' S^{\text{sp conf}}_{(V)} = -ib_0\int du d\zeta^{--}\,\frac{\delta'\hat{V}^{++}}{\left(\sqrt{1+c^{--}\hat{V}^{++}}\right)^3}\,. \tag{5.33}$$

Following (5.11), the half-potential and vector potential specifying the bosonic sector of (5.32) are given by the following harmonic integrals,

$$\mathcal{V}^{\rm conf} = \int du \frac{1}{\left(\sqrt{1+c^{--}\hat{v}^{++}}\right)^3}\,, \tag{5.34}$$

$$\mathcal{A}^{\rm conf}_{ik} = 2\int du\, \frac{u^+_{(i}u^-_{k)}}{\left(\sqrt{1+c^{--}\hat{v}^{++}}\right)^3} \tag{5.35}$$

(b_0 was identified here with the overall normalization constant γ' and detached). The harmonic integral (5.34) already appeared (in another context) in [30] and was computed there. Using this result we find

$$\mathcal{V}^{\rm conf} = \frac{\sqrt{2}}{\sqrt{v^{ik}v_{ik}}} = \frac{\sqrt{2}}{|\mathbf{v}|}\,. \tag{5.36}$$

We notice that the constant triplet c^{ik} could be absorbed into $v^{ik} = \hat{v}^{ik} + c^{ik}$ after the u-integration was performed, indicating that the c^{ik} are moduli of the theory. Yet, they still appear explicitly in $\mathcal{A}^{\rm conf}_{ik}$. The latter can be calculated by choosing in (5.35) an explicit parametrization of harmonics, e.g. by Euler angles, and performing the integral over S^2. However, it is simpler to directly restore $\mathcal{A}^{\rm conf}_{ik}$ from the general constraint (5.11):

$$\mathcal{A}^{\rm conf}_{ik} = -\sqrt{2}\frac{c^p_i v_{pk} + c^p_k v_{pi}}{[(\mathbf{v}\cdot\mathbf{c}) + \sqrt{2}|\mathbf{v}|]|\mathbf{v}|}\,. \tag{5.37}$$

This is the potential of a Dirac magnetic monopole, with c^{ik} parametrizing the singular Dirac string. Thus, in the manifestly $N=4$ supersymmetric formulation of the conformally invariant superpotential these parameters arise already at the superfield lagrangian level. However, the complete action does not depend on them, because of its scale invariance and its invariance under the 'conformal' SU(2) (the one which acts on the indices i,j). The same is valid for the bosonic part of the action: when (5.37) is substituted into (5.10), all terms with manifest c-dependence are reduced to a total t-derivative [19]. The bosonic sector of $L^{++}_{\rm conf}$ precisely coincides with that derived in [19] from the formulation of $N=4$ superconformal mechanics in the standard $N=4$, $d=1$ superspace and in terms of $N=2$ superfields.

It is worth pointing out once more that the possibility to write the superpotential term in manifestly $N=4$ supersymmetric form (both for the generic prepotential L^{++} and for the superconformally invariant one $L^{++}_{\rm conf}$) is offered only by the analytic harmonic $N=4$, $d=1$ superspace. One can of course rewrite them as integrals over the complete $N=4$ superspace, either in terms of the prepotential for V^{ik} or with explicit θ s, like this has been given for the superconformal case in [19]. However, such a representation, as opposed to the formulation in $\mathbf{AR}^{1+2|2}$, lacks manifest supersymmetry (or makes one to care about some superfluous gauge invariances) and does not suggest any hint how to generalize the superconformally invariant superpotential (or the FI term) to the generic case.

5.4 Further examples of superpotentials

Note that the general spherically symmetric half-potential (5.17) also yields the expression (5.37) for $\mathcal{A}^{\rm conf}_{ik}$, but for any non-zero constant c_0 it breaks conformal invariance, and the corresponding L^{++} does not respect $N=4$ superconformal symmetry. Such a superpotential is obtained by adding to $L^{++}_{\rm conf}$ the non-conformal piece $c_0 V^{++}$, i.e.

$$L^{++}_{TN} = c_0\, V^{++} + \frac{1}{\sqrt{2}}\, L^{++}_{\rm conf}\,. \tag{5.38}$$

Though L^{++}_{TN} breaks the whole $N=4$ superconformal group, it is still invariant under the SU(2) subgroup of the latter which affects the harmonics and the doublet indices of the original $\theta_i, \bar\theta^k$ and acts as rotations on v^{ik}. It corresponds to the choice

$$\Lambda_{su(2)} = \lambda^{(ik)} u^+_i u^-_k = \frac{1}{2} D^{++} \lambda^{(ik)} u^-_i u^-_k \tag{5.39}$$

in the transformation laws (3.18), (5.21). Varying the first term in (5.38) as $\delta_{su(2)} V^{++} = 2\Lambda_{su(2)} V^{++}$, using (5.39) and integrating by parts with respect to D^{++}, one proves the SU(2) invariance of this term. The second term is invariant under the whole $D(2,1;\alpha)$ and hence under any of its subgroups.

The superpotential (5.38) is a $d=1$ analog of one of the two forms of the Taub-NUT sigma-model lagrangian in analytic harmonic $N=2$, $d=4$ superspace [21]: both are specified by the Taub-NUT potential (5.17).[9] It is interesting to find the superpotential producing the well known two-center potential for the Eguchi-Hanson metric as the associated half-potential $\mathcal{V}$. Such an L^{++}_{EH} is given by

$$\begin{aligned} L^{++}_{EH} = {} & \frac{c_1\, V^{++}}{\left(1+\sqrt{1-V^{++}a^{--}}\right)\sqrt{1-V^{++}a^{--}}} + \\ & + \frac{(V^{++}-c^{++})}{\left(1+\sqrt{1+(V^{++}-c^{++})c^{--}}\right)\sqrt{1+(V^{++}-c^{++})c^{--}}}\,, \end{aligned} \tag{5.40}$$

where $a^{--} = a^{ik} u^-_i u^-_k\,,\ \ a^{ik} a_{ik} = 2$, and it leads to

$$\mathcal{V}_{EH} = \int du\, \frac{1}{(\sqrt{1+(v^{++}-c^{++})c^{--}})^3} + c_1 \int du\, \frac{1}{(\sqrt{1-v^{++}a^{--}})^3}\,. \tag{5.41}$$

These harmonic integrals are computed like in the previous case, yielding, up to an overall factor,

$$\mathcal{V}_{EH} = \frac{1}{|\mathbf{v}|} + c_1 \frac{1}{|\mathbf{v}-\mathbf{a}|}\,. \tag{5.42}$$

This half-potential has poles located at $\mathbf{v}=0\,,\ \mathbf{v}=\mathbf{a}$ and, up to normalization and shifts of v^{ik}, coincides with the standard potential for the Eguchi-Hanson metric [40]. Adding to (5.40) a term $\sim V^{++}$ adds a constant to (5.42) and so produces a potential for what is called the 'double Taub-NUT metric' [40].

[9]To avoid a possible misunderstanding, let us point out that in the former case one deals with a 4-dimensional hyper-Kähler manifold possessing SU(2) × U(1) isometry group. In the multi-center ansatz the U(1) isometry is realized as a shift of 4th coordinate and affects neither the Taub-NUT potential nor the relevant vector potential which are defined on $\mathbb{R}^3$. In our case we are dealing at once with a 3-dimensional bosonic manifold and the Taub-NUT scalar and vector potentials defined on it; so the maximal invariance group of (5.38) is SU(2).

5.5 Superpotentials in the $3n$-dimensional case

It is straightforward to generalize (5.9) and (5.10) to the case with n superfields V^{++}_A, $A = 1, 2, \ldots, n$, via

$$S^{n\ sp}_{(V)} = -i\gamma' \int dud\zeta^{--} L^{++}(V^{++}_A, u)\,. \tag{5.43}$$

Its bosonic part reads

$$\hat{S}^{n\ sp}_{(V)} = \gamma' \int dt \left\{ F_B\, \mathcal{V}^B(v) + \dot{v}^{ik}_B\, \mathcal{A}^B_{ik}(v) \right\}, \tag{5.44}$$

with

$$\mathcal{V}^B(v) = \int du\, \frac{\partial L^{++}}{\partial v^{++}_B}\,, \quad \mathcal{A}^B_{ik}(v) = 2\int du\, u^+_{(i} u^-_{k)}\, \frac{\partial L^{++}}{\partial v^{++}_B}\,. \tag{5.45}$$

The full potential can be read off by integrating out the auxiliary fields F_B in the sum of the actions (2.16) and (5.44). Note that in the general case this procedure modifies as well the coupling to the external gauge potential because of a mixed term $\sim F_A \dot{v}^{ik}_B$ in (2.16).

The constraints generalizing (5.11) again follow from the explicit expressions (5.45),

$$\begin{aligned} \partial^A_{ik}\mathcal{A}^B_{lt} - \partial^B_{lt}\mathcal{A}^A_{ik} &= \epsilon_{il}\, \partial^A_{kt}\mathcal{V}^B + \epsilon_{kt}\, \partial^A_{il}\mathcal{V}^B\,, \\ \Delta^{CD}\mathcal{V}^A &= 0\,, \qquad \partial^{[A}_{ik}\mathcal{V}^{B]} = 0\,. \end{aligned} \tag{5.46}$$

These constraints resemble those arising in the ansatz for a toric $4n$-dimensional hyper-Kähler metric [41] which generalizes the Gibbons-Hawking 4-dimensional one. The difference lies however in the fact that our half-potential $\mathcal{V}^A$ and one-form $\mathcal{A}^A_{ik}$ are vectors with respect to the extra index A, while in the ansatz of [41] the analogous objects are rank 2 symmetric tensors.

Finally, we would like to point out that the properties of the sigma-model and superpotential parts of the full action for V^{ik} do not correlate: in general, their only common property is $N = 4$ supersymmetry. For instance, one can choose the scalar half-potential and one-form potential to be spherically symmetric as in the examples above and, at the same time, not assume such a symmetry for the metric functions in (2.14), (2.16), or vice versa.

6. The actions for $\mathbf{q}^+$

6.1 General q^+ sigma-model action

The constrained analytic $N = 4$ superfield q^{+a} is defined by eqs. (4.13)-(4.18). The general sigma-model type off-shell action for q^{+a}, like in the case of V^{++}, can be written either as an integral over $\mathbb{R}^{1|4}$ of a function of the ordinary constrained superfields $q^{ia}(z)$ (see (4.18)), or, equivalently, as an integral over $\mathbf{HR}^{1+2|4}$ of a function of the q^{+a} with $q^{-a} = D^{--}q^{+a}$:

$$S_{(q)} = -\int dtd^4\theta\, L(q) = -\int du\, dtd^4\theta\, L'(q^+, q^-, u)\,,\ L(q) = \int du\, L'(q^+, q^-, u)\,. \tag{6.1}$$

The simplest way to compute the bosonic sigma-model action is to find the bosonic part of q^{ia} (with fermions omitted) and substitute it into (6.1). Passing to the central basis in the explicit expression (4.15) for $q^{+a} = q^{ia}(z)u^+_i$ and suppressing the fermions, we find

$$q^{ia}(z) = f^{ia}(t) - i\dot{f}^{ka}(t)(\theta_k\bar{\theta}^i - \bar{\theta}_k\theta^i) + \frac{1}{4}\ddot{f}^{ia}(t)\theta^2\bar{\theta}^2 \,. \tag{6.2}$$

Then, for the bosonic core of (6.1) we obtain the simple expression

$$\hat{S}_{(q)} = \int dt\, \Delta L\, (\dot{f}\cdot\dot{f})\,, \tag{6.3}$$

where now $\Delta = \partial^2/\partial f\cdot\partial f$ and "$\cdot$" denotes contraction over the $\mathbb{R}^4$ indices, $x\cdot y \equiv x^{ia}y_{ia}$. So we encounter a conformally-flat geometry in $\mathbb{R}^4$. An extension to the case of several superfields q^{+a}_A, $A = 1,\ldots,n$, is straightforward. One changes $L(q) \to L(q_1, q_2, \ldots)$ and obtains the bosonic action in the form

$$\hat{S}^{\,n}_{(q)} = -2\int dt\, g^{AB}_{a\,b}\,(\dot{f}^{ja}_A\,\dot{f}^{b}_{j\,B})\,, \qquad g^{AB}_{a\,b} = \epsilon^{tj}\frac{\partial^2 L}{\partial f^{ta}_A\,\partial f^{jb}_B} \tag{6.4}$$

The general characterization of this geometry was given in [10, 16, 32, 8]. As found in [10, 8] (see also [42]), this geometry is in general weak HKT and for special choices of the potentials $L(f)$ can be strong HKT (torsion appears in the fermionic terms).

Like in the case of V^{++}, one can inquire whether for some special $L(q)$ the above actions admit a representation as integrals over the analytic harmonic superspace $\{\zeta, u\}$ with manifestly analytic superfield lagrangians. Let us start from the free theory which corresponds to the choice $L(q) = \frac{1}{8}q\cdot q$

$$S^{\rm free}_{(q)} = -\frac{1}{8}\int dt d^4\theta\,(q\cdot q) = -\frac{1}{4}\int du\, dt d^4\theta\, q^{+a}D^{--}q^+_a \,. \tag{6.5}$$

Passing, in the second form of this action, to the integral over the analytic superspace with the help of (3.24) and (3.17), we find a very simple analytic superspace form of this kinetic term,

$$S^{\rm free}_{(q)} = \frac{i}{2}\int du d\zeta^{--}\,(q^{+a}\dot{q}^+_a) \qquad \Longrightarrow \qquad \hat{S}^{\rm free}_{(q)} = \int dt\,(\dot{f}^{ia}\dot{f}_{ia})\,. \tag{6.6}$$

One can immediately generalize it to the interaction case. For several q^{+a}_A, this generalization reads

$$S^{n\,\sigma}_{(q)} = i\gamma\int du d\zeta^{--}\,\mathcal{L}^{+aA}(q^+, u)\,\dot{q}^+_{aA}\,, \tag{6.7}$$

where $\mathcal{L}^{+aA}(q^+, u)$ is some unconstrained analytic prepotential, depending on q^{+a}_A and explicit harmonics. It is very easy to obtain the bosonic action,

$$\hat{S}^{n\,\sigma}_{(q)} = -2\gamma\int dt\left(\int du\frac{\partial\mathcal{L}^{+A}_a}{\partial f^{+b}_B}\right)\dot{f}^{ka}_A\,\dot{f}^b_{k\,B}\,. \tag{6.8}$$

Comparing with (6.4) we find (up to the normalization γ)

$$\tilde{g}^{AB}_{a\,b} = \int du\,\frac{\partial\mathcal{L}^{+A}_a}{\partial f^{+b}_B}\,, \quad \tilde{L}(f) = \int du\,\mathcal{L}^{+aA}q^-_{a\,A}\,. \tag{6.9}$$

where the tilde indicates that we deal with a particular case of (6.4). It immediately follows from (6.9) that such $\tilde{g}^{AB}_{a\,b}$ — as a main specific feature — satisfy the generalized harmonicity condition

$$\Delta^{CD}\tilde{g}^{AB}_{a\,b} = 0 \qquad \text{with} \qquad \Delta^{CD} = \frac{\partial^2}{\partial_C \cdot \partial_D}\,. \tag{6.10}$$

In the 4-dimensional case ($n = 1$) only the piece $\sim \epsilon_{ab}$ in $\tilde{g}_{a\,b}$ contributes to (6.8), and the latter takes the conformally-flat form

$$\hat{S}^{1\,\sigma}_{(q)} = -\gamma \int dt\, H(f)\,(\dot{f}\cdot\dot{f})\,, \qquad H(f) = \int du \frac{\partial \mathcal{L}^{+a}}{\partial f^{+a}}\,, \qquad \Delta H(f) = 0\,. \tag{6.11}$$

This metric belongs to the class exhibited in [43, 32], and it defines a strong HKT geometry associated with flat $\mathbb{R}^4$ as the relevant hyper-Kähler manifold (in order to see this one should examine the torsion which appears in the fermionic terms of (6.7)).

Thus, similarly to the V^{++} case, we observe an interesting phenomenon. Requiring off-shell $N = 4$ supersymmetric sigma-model actions for q^{+a} to be representable in the analytic harmonic superspace, with the lagrangians being local functions of analytic q^{+a} superfields, imposes further restrictions on the target-space geometry of such $d = 1$ sigma models, in addition to those which are already implied by $N = 4$ supersymmetry and are automatically satisfied in the superfield approach.

6.2 $N = 4$ supersymmetric off-shell coupling to the background one-form

One can easily write the analog of the superpotential (5.9) for the field q^{+a},

$$S^{\rm sp}_{(q)} = -\frac{i}{2}\int du d\zeta^{--}\, \mathcal{L}^{++}(q^+, u)\,. \tag{6.12}$$

However, since the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet does not involve any auxiliary field, (6.12) does not give rise to any scalar potential and only provides a coupling to the external background one-form. It is the most general manifestly $N = 4$ supersymmetric invariant of the appropriate dimension. One can expect that the linearized version of the constraints on the background one-form found in [16] by *requiring* $N = 4$ supersymmetry automatically emerges from (6.12).[10]

For the simplest case of a single flied q^{+a} one gets the following bosonic action,

$$\hat{S}^{\rm sp}_{(q)} = \int dt\, \dot{f}^{ia}\,\mathcal{A}_{ia}(f)\,, \qquad \mathcal{A}_{ia}(f) = \int du\, u^-_i \frac{\partial \mathcal{L}^{++}}{\partial f^{+a}}\,. \tag{6.13}$$

From the explicit representation for $\mathcal{A}_{ia}$ it is easy to check that

$$\mathcal{F}_{kb\,ia} := \partial_{kb}\mathcal{A}_{ia} - \partial_{ia}\mathcal{A}_{kb} = \epsilon_{ki}\int du\, \frac{\partial^2 \mathcal{L}^{++}}{\partial f^{+a}\,\partial f^{+b}} =: \epsilon_{ki}\mathcal{B}_{(ab)}\,, \tag{6.14}$$

which is a self-duality condition for the $\mathbb{R}^4$ abelian gauge field. The same condition has been obtained in [16]. In addition, (6.13) implies that

$$\partial^{ia}\mathcal{A}_{ia} = 0\,, \tag{6.15}$$

[10]The authors of [16] considered some nonlinear version of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet constraints. For the moment we do not know how to describe such a nonlinear $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet in harmonic superspace.

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which can be interpreted as a choice of transversal gauge for $\mathcal{A}_{ia}$. Together with the Bianchi identity

$$\partial_i{}^b \mathcal{B}_{(ba)} = 0 \tag{6.16}$$

the self-duality condition (6.14) implies, as usual, that

$$\partial^{ia} \mathcal{F}_{ia\, kb} = 0 \tag{6.17}$$

which, with taking account of (6.15), yields

$$\Delta \mathcal{A}_{ik} = 0\,. \tag{6.18}$$

This can be checked directly using (6.13). The constraints (6.14) and (6.15) are the linearized (adapted to the linear $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet) form of the constraints derived in [16].

It is straightforward to generalize this consideration to several superfields q^{+a}_C. One should then allow for extra q^{+a}_B superfields in $\mathcal{L}^{++}$ in (6.12). The relevant one-form potential is given by the harmonic integral

$$\mathcal{A}^C_{ia} = \int du\, u^-_i \frac{\partial \mathcal{L}^{++}}{\partial f^{+a}_C}\,. \tag{6.19}$$

It satisfies the following analogs of constraints (6.14), (6.15):

$$\mathcal{F}^{CD}_{ia\,kb} := \partial^C_{kb}\mathcal{A}^D_{ia} - \partial^D_{ia}\mathcal{A}^C_{kb} = \epsilon_{ki} \int du\, \frac{\partial^2 \mathcal{L}^{++}}{\partial f^{+a}_D\, \partial f^{+b}_C} =: \epsilon_{ki}\mathcal{B}^{CD}_{ab}\,, \tag{6.20}$$

$$\partial^{(B} \cdot \mathcal{A}^{C)} \;=\; 0\,, \tag{6.21}$$

as well as the generalized harmonicity condition

$$\Delta^{AB}\mathcal{A}^C_{ia} = 0\,. \tag{6.22}$$

The last remark concerns possible scalar potentials of the fields f^{ia}. As already mentioned, in the pure q^{+a} models such terms cannot appear because of absence of auxiliary fields in the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet. However, they can appear in mixed systems, e.g. with both q^{+a} and V^{++} superfields involved. The simplest possibility of this sort is to consider the manifestly $N=4$ supersymmetric analytic superpotential

$$S_{(Vq)} = -i \int dud\zeta^{--}\, V^{++}\, L(q^+, u)\,, \tag{6.23}$$

which in the bosonic sector yields the term

$$\hat{S}_{(Vq)} = \int dt\; F(t)\, \left(\int du\; L(f^+, u) \right) \quad + \quad \text{coupling to a one-form potential}\,. \tag{6.24}$$

The half-potential $\mathcal{V}^{(q)}(f) = \int duL(f^{+a}, u)$ satisfies the four-dimensional Laplace equation. After $F(t)$ is integrated out from the sum of (6.24) and the free V^{ik} action (5.3), one ends up with a scalar potential which depends only on f^{ia}. If instead of the free V^{ik} action we take the general sigma-model type action (2.13), the eventual scalar potential will be a function of both v^{ik} and f^{ja}.

One further possibility is to include into the game the odd-Grassmann parity version of the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet. It is described off shell by a fermionic analytic superfield $\Psi^{+a}(\zeta,u)$ satisfying the harmonic constraint

$$D^{++}\Psi^{+a} = 0\,. \tag{6.25}$$

Therefore, it has the same $\theta^+,\bar\theta^+$ expansion as q^{+a} (4.15), namely

$$\Psi^{+a}(\zeta,u) = \psi^{ia}u^+_i + \theta^+\xi^a + \bar\theta^+\bar\xi^a + 2i\,\theta^+\bar\theta^+\dot\psi^{ia}u^-_i\,, \tag{6.26}$$

with the only (but essential) difference that ψ^{ia} are now physical fermions while $\xi^a,\bar\xi^a$ form a complex doublet of bosonic auxiliary fields. So it can be called the $(\mathbf{0},\mathbf{4},\mathbf{4})$ multiplet. The free action of it has the following nice form,

$$S^{\rm free}_{(\psi)} = \frac{1}{2}\int du d\zeta^{--}\;\Psi^{+a}\Psi^+_a = \int dt\left(i\psi^{ia}\dot\psi_{ia} + \xi^a\bar\xi_a\right). \tag{6.27}$$

When appropriately coupled to q^{+a}, this multiplet is also presumably capable to produce scalar potentials for bosons f^{ia} upon eliminating the auxiliary fields $\xi^a,\bar\xi^a$.

6.3 Superconformally invariant q^+ actions

It is natural to start the investigation of $D(2,1;\alpha)$ invariant actions for q^{+a} with the free action (6.5). As before we shall be interested in invariance under the transformations of conformal supersymmetry. Taking into account the $D(2,1;\alpha)$ transformation law (4.17) and the fact that $\partial/\partial t_A$ is properly transformed through partial derivatives in the analytic Grassmann coordinates, the superconformal variation of (6.5) is as follows (up to a total D^{++} derivative),

$$\delta' S^{\rm free}_{(q)} = -2(1-\alpha)\int du d\zeta^{--}\,(\epsilon^-\bar\theta^+ - \bar\epsilon^-\theta^+)\,q^{+a}\dot q^+_a\,. \tag{6.28}$$

We see that the free action is only invariant provided that $\alpha = 1$.

Nevertheless, one can define nonlinear sigma-model type $D(2,1;\alpha)$ invariant actions for q^{+a} for any value of α. Like in the case of V^{++}, such actions admit a field theoretic interpretation (i.e. contain a kinetic part) only under the assumption that the bosonic fields f^{ia} start with some constant, $f^{ia} = \epsilon^{ia} + \ldots$. This means that $N=4$ superconformal symmetry is spontaneously broken and q^{+a} is the corresponding Goldstone superfield, analogous to the interpretation of V^{++} in [19]. This issue is considered in [12]. Here we wish to show that such superconformal actions can be constructed without any reference to the nonlinear realization formalism used in [19, 12].

The idea behind the whole construction is very simple. Let us define a composite $N=4$ analytic superfield

$$\mathcal{V}^{++} = q^{+a}a_{ab}q^{+b}\,, \tag{6.29}$$

where a_{ab} is a constant symmetric tensor which breaks the extra Pauli-Gürsey SU(2) (realized on the indices a,b) down to some U(1) subgroup. We choose

$$a^2 = a^{ab}a_{ab} = 1\,. \tag{6.30}$$

The superfield (6.29) possesses all the properties of V^{++}. Indeed, it satisfies

$$D^{++}\mathcal{V}^{++} = 0\,, \tag{6.31}$$

as a consequence of the q^{+a} defining constraint (4.14), and it transforms under $D(2,1;\alpha)$ as

$$\delta'\mathcal{V}^{++} = 2\lambda\mathcal{V}^{++} \tag{6.32}$$

as a result of the q^{+a} transformation law (4.17). Hence, any action of V^{++}, including the superconformally invariant ones (2.18), (2.19), upon substitution $V^{++} \to \mathcal{V}^{++}$ will produce an $N = 4$ supersymmetric action for q^{+a}. This substitution applied to (2.18), (2.19) evidently produces the desired $D(2,1;\alpha)$ invariant actions for q^{+a}. Using the central basis form of (6.29),

$$\mathcal{V}^{ik}(z) = q^{ia}(z)a_{ab}q^{kb}(z)\,, \tag{6.33}$$

and the relation

$$\mathcal{V}^2 = \frac{1}{4}(q^2)(q^2) \qquad \text{where} \quad q^2 \equiv q^{ia}q_{ia}\,, \tag{6.34}$$

it is easy to recover the superconformally invariant actions of q^{+a} as

$$S^{\text{conf}}_{(q)}(\alpha) = \left\{ \begin{array}{ll} -\gamma\int dtd^4\theta\;(q^2)^{1/\alpha} & \text{for} \quad \alpha \neq -1 \\ \gamma'\int dtd^4\theta\;(q^2)^{-1}\ln(q^2) & \text{for} \quad \alpha = -1 \end{array} \right. . \tag{6.35}$$

At $\alpha = 1$ the upper version becomes free and we reproduce the previous result. On the other hand, the free action of V^{ik} corresponding to $\alpha = 1/2$ in (2.18) produces a nonlinear sigma-model type action for q^{ia} upon the substitution (6.33).

The component bosonic action for (6.35) is easy to compute by the general formula (6.3):

$$\hat{S}^{\text{conf}}_{(q)}(\alpha) = \left\{ \begin{array}{ll} \gamma\frac{1+\alpha}{\alpha^2}\displaystyle\int dt\,(f^2)^{\frac{1-\alpha}{\alpha}}\,(\dot{f}\cdot\dot{f}) & \text{for} \quad \alpha \neq -1 \\ \gamma'\int dt\,(f^2)^{-2}\,(\dot{f}\cdot\dot{f}) & \text{for} \quad \alpha = -1 \end{array} \right. . \tag{6.36}$$

Except for the free $\alpha = 1$ case, they make sense only under the assumption that the 'vacuum value' of the radial part of q^{ia} is non-vanishing, i.e. $\langle q^2\rangle \neq 0$. This means that the deviation $\tilde{q}^2 = q^2 - \langle q^2\rangle$ is a dilaton with an inhomogeneous transformation law. Thus, dilatation invariance is spontaneously broken. Analogously, one can see that the whole R-symmetry $\mathrm{SU}(2)_R$ acting on the indices i,j and rotating the harmonics is spontaneously broken, with the angular part of q^{ia} being the corresponding Goldstone fields. This is to be contrasted with the field V^{ik} which also involves the dilaton as its radial part, but breaks $\mathrm{SU}(2)_R$ only down to U(1) [19]. In both cases, another SU(2) R-symmetry present in $D(2,1;\alpha)$ is linearly realized on the physical fermions and so is unbroken. The superfield q^{ia}, similarly to V^{ik}, can be derived from the appropriate nonlinear realization of $D(2,1;\alpha)$ as a Goldstone superfield [44]. It is interesting that the Pauli-Gürsey SU(2) group acting on the index a of q^{ia} and explicitly broken by the ansatz (6.29) is actually restored in the superconformal action (6.35). The latter is thus manifestly SO(4) invariant.

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For later use, it is of interest to recognize how the component fields of $\mathcal{V}^{++}$ are expressed in terms of those of q^{+a}. Substituting the component expressions (4.4), (4.15) into (6.29), we obtain

$$\mathcal{V}_0^{ik} = f^{ia}a_{ab}f^{kb}\,, \quad \mathcal{F} = 2\,(\,\dot{f}_i{}^a a_{ab}f^{ib} + i\chi^a a_{ab}\bar{\chi}^b\,)\,, \quad \psi^i = 2\chi^a a_{ab}f^{ib}\,, \quad \bar{\psi}^i = 2\bar{\chi}^a a_{ab}f^{ib}\,. \tag{6.37}$$

The property that the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplet can be obtained from the $(\mathbf{3},\mathbf{4},\mathbf{1})$ one by replacing the auxiliary field in the latter by a time derivative of some new scalar $d=1$ field was mentioned in [33] as a particular case of a more general phenomenon. The relations (6.29), (6.37) can be regarded as a nonlinear version of this correspondence.

We now can study what kind of the superpotential-type invariant for q^{+a} the substitution (6.29) produces from the superconformal superpotential (5.32). The direct insertion of the expressions (6.37) into (5.10) with $\mathcal{V}_{\rm conf}$ and $\mathcal{A}^{\rm conf}_{ik}$ given by (5.34) and (5.35) yields, up to a normalization factor,

$$S^{sp\ conf}_{(q)} \implies \int dt \dot{f}^{ia}\mathcal{A}^{\rm conf}_{ia}(f)\,, \tag{6.38}$$

where

$$\mathcal{A}^{\rm conf}_{ia}(f) = \frac{1}{[f^2 + 2(f\cdot a\cdot f\cdot c)]}\left(f^k_a c_{ki} - f^b_i a_{ab}\right) \tag{6.39}$$

and

$$f\cdot a\cdot f\cdot c = f^{ia}f^{kb}c_{ik}a_{ab}\,.$$

However, calculating the curl of this vector potential, one finds the disappointing result

$$\partial_{kb}\mathcal{A}^{\rm conf}_{ia} - \partial_{ia}\mathcal{A}^{\rm conf}_{kb} = 0 \tag{6.40}$$

i.e. $\mathcal{A}^{\rm conf}_{ia}$ is pure gauge and (6.38) is a total derivative. Note that the crucial role in achieving this negative result is played by the term with the auxiliary field in (5.10): now it is proportional to $\dot{f}^{ia}$, and its contribution combines with that from the magnetic coupling in (5.10) to produce the total derivative. An inspection of the fermionic terms in (5.32) with a composite V^{++} also shows that they vanish: the term $\sim \psi^i\bar{\psi}^k$ after the substitution (6.37) is cancelled out by a similar term coming from the 'auxiliary field' $\mathcal{F}$.

Thus, rather surprisingly, an $N=4$ superconformally invariant coupling of q^{ia} to the one-form target-space potential does not exist. For the time being we do not fully understand what stands behind this property. Presumably, it is related to the absence of WZW terms for the fully non-linearly realized SU(2) group for which the angular part of the $\mathbb{R}^4$ vector f^{ia} provides a parametrization [12]. In the $\mathbb{R}^3$ case one deals with the coset SU(2)/ U(1), and the $d=1$ WZW term associated with this U(1) is just the conformally invariant coupling of v^{ik} to the magnetic monopole [36, 45, 19].

6.4 Isometries

One may ask what is the characteristic feature of the subclass of q^+ actions with which we end up after substituting the composite superfield $\mathcal{V}^{ik}$ (or $\mathcal{V}^{++}$) into a general V^{++} action. The answer is that such a subclass is distinguished by its U(1) symmetry down to which the constant vector a_{ab} breaks the Pauli-Gürsey SU(2) and under which (6.29), (6.33) and (6.37) are manifestly invariant. This U(1) is an isometry of the relevant target-space metric and one-form potential. In the case of n superfields q^{+a}_B one can define n composite superfields $\mathcal{V}_{ia\,A}$ via the recipe (6.29) with n independent constant tensors a_{ab}. Substituting such composite superfields into the generic V^{++}_B action gives rise to the subclass of q^+ actions which yield n commuting U(1) isometries of the rotational type.

Like in the case of $N=2$, $d=4$ hypermultiplets in harmonic superspace [21], one can define on q^{+a} also an isometry of the translational type,

$$q^{+a\prime} = q^{+a} + \lambda u^{+a} \quad \text{or} \quad q^{ia\prime} = q^{ia} + \lambda \epsilon^{ai}\,. \tag{6.41}$$

The defining constraint (4.14) is evidently invariant under (6.41). Projecting q^{+a} on the harmonics via

$$l^{++} \equiv q^{+a}u^+_a \quad \omega \equiv q^{+a}u^-_a\,, \tag{6.42}$$

one finds that

$$l^{++\prime} = l^{++}\,, \quad \omega' = \omega + \lambda\,, \tag{6.43}$$

so the q^{+a} lagrangians which respect this isometry are characterized by their independence of ω. Observing that the defining q^{+a} constraint (4.14) implies

$$D^{++}l^{++} = 0\,, \tag{6.44}$$

we conclude that l^{++} is again a composite $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplet with

$$l^{ik} = \delta^{(i}_a f^{k)\,a}\,, \qquad \psi^i = \delta^i_a \chi^a\,, \qquad f = \epsilon_{ai}\dot{f}^{ia}\,. \tag{6.45}$$

This linear type of correspondence between the multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$ is just the one discussed in [33]. General q^+ actions possessing this type of isometry can be obtained by substituting l^{++} for V^{++} into generic actions for the latter, discussed in section 5.1. At the level of bosonic actions, this can be rephrased: Imposing invariance under the isometry (6.41) forces the general f^{ia} actions (6.3), (6.13) to coincide with those for $v^{ik} = l^{ik}$, with the time derivative of the fourth coordinate $\epsilon_{ai}\dot{f}^{ia}$ mimicking the auxiliary field F. This agrees with the reduction procedure from the sigma models based on $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets to those with $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplets, as it was described in [16]. Obviously, this works for any number of q^{+a}_A multiplets.

Finally, we consider two examples.

Requiring $L(q)$ in (6.1) to depend only on the symmetric combination $q^{(ia)}$ and so be invariant under (6.41) gives us the following particular case of an $\mathbb{R}^4$ bosonic sigma-model action (6.3):

$$\hat{S}^{\rm red}_{(q)} = \int dt\, \Delta L(\mathbf{f}) \left(\dot{\mathbf{f}}\cdot\dot{\mathbf{f}} + \frac{1}{2}\dot{f}\dot{f}\right), \qquad \Delta = \frac{\partial^2}{\partial\mathbf{f}}\cdot\partial\mathbf{f}\,. \tag{6.46}$$

This should be compared with the general bosonic v^{ik} action (2.14).

As another example, let us consider the subclass of q^{+a} superpotentials (6.12) invariant under (6.41):

$$S^{1\ sp}_{(q)} = -\frac{i}{2}\int du d\zeta^{--}\mathcal{L}^{++}(q^+u^+, u)\,. \tag{6.47}$$

The corresponding one-form potential defined in (6.13) takes the form

$$\mathcal{A}_{ia}(\mathbf{f}) = \int du\, u_i^- u_a^+ \frac{\partial \mathcal{L}^{++}}{\partial (f^+u^+)}\,. \tag{6.48}$$

Splitting it into the $\mathbb{R}^3$ and fourth components,

$$\mathcal{A}_{(ia)}(\mathbf{f}) = \int du\, u^-_{(i} u^+_{a)} \frac{\partial \mathcal{L}^{++}}{\partial (f^+u^+)}\,, \qquad \mathcal{A}_4(\mathbf{f}) = \epsilon^{ia}\mathcal{A}_{ia} = \int du\, \frac{\partial \mathcal{L}^{++}}{\partial (f^+u^+)}\,, \tag{6.49}$$

we see that they coincide with the $\mathbb{R}^3$ one-form potential and scalar half-potential (5.11) of the general superpotential for the superfield V^{++} (with $v^{++} = f^{ia}u_i^+u_a^+$). The constraints (5.13) are recognized as the 'static' monopole ansatz for the $\mathbb{R}^4$ self-duality equation (6.14). The general bosonic $f/\mathcal{A}$ coupling (6.13) for this particular case takes the form

$$\hat{S}^{1\ sp}_{(q)} = \int dt \left(\dot{\mathbf{f}}\cdot \mathcal{A} + \frac{1}{2}\dot{f}\mathcal{A}_4 \right). \tag{6.50}$$

The SO(4) symmetry is clearly broken in this action. An $\mathbb{R}^4$ particle with this type of interaction can be interpreted as a dyonic particle: the 4th component of its position field is coupled to a static electric potential while the $\mathbb{R}^3$ component sees a static magnetic field. A particular SO(3) invariant example describes a coupling to the Coulomb potential (5.36) and the Dirac monopole potential (5.37). Note that the corresponding $N = 4$ supersymmetric lagrangian is obtained by performing the substitution $\hat{V}^{++} \to (q^{+a}u_a^+) - c^{++}$ in the superconformally invariant action (5.32). The resulting q^{+a} action, however, is not superconformally invariant, since the transformation law of $l^{++} = q^{+a}u_a^+$ under $D(2,1;\alpha)$ is different from that of V^{++},

$$\delta' l^{++} = \Lambda\, l^{++} + \Lambda^{++}\,\omega\,. \tag{6.51}$$

Finally, we note that for the q^{+a} actions the geometries of the sigma-model part and those of the superpotential part do not correlate with each other like in the V^{++} case, and so one can impose invariance requirements on these pieces separately. Typically the full actions then are not obliged to respect any symmetry except for $N = 4$, $d = 1$ Poincaré supersymmetry.

7. Concluding remarks

In this paper we have introduced the $N = 4$, $d = 1$ harmonic superspace as a new setting for SQM models with $N = 4$ supersymmetry. We have shown that the off-shell $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets, which were utilized earlier for $N = 4$ SQM model building in the framework of standard $N = 4$, $d = 1$ superspace, have a natural description as constrained analytic harmonic superfields $V^{++}(\zeta,u)$ and $q^{+a}(\zeta,u)$, respectively. The analytic harmonic superspace was shown to be closed under the most general $N = 4$, $d = 1$ superconformal group $D(2,1;\alpha)$ at any value of the parameter α, and the realization of this supergroup on V^{++} and q^{+a} was found. We presented harmonic superspace actions for these superfields, both in the superconformal and in the generic cases, and demonstrated that the conditions on the bosonic target-space metrics, scalar potentials and one-forms required by $N = 4$ supersymmetry are automatically reproduced from this manifestly $N = 4$ supersymmetric off-shell description. The superpotential-type pieces of the full action are given by integrals over the $(1 + 2|2)$-dimensional analytic subspace of $N = 4$, $d = 1$ harmonic superspace, and it is the only possibility to write down such terms off shell in a manifestly $N = 4$ supersymmetric manner.

As one problem for future study, it is tempting to construct and examine SQM models associated with more general superfields $q^{(+n)}(\zeta, u)$ subject to $D^{++}q^{(+n)} = 0$ for $n > 2$, which are suggested by the harmonic superspace approach. Some other consrained analytic multiplets from $N = 2$, $d = 4$ harmonic superspace [21] also have d= 1 analogs. For instance, one can define a nonlinear multiplet $N^{++}(\zeta, u)$ by the constraint $D^{++}N^{++} + (N^{++})^2 = 0$. Although this $N = 4$, $d = 1$ constraint has the same form as in the $N = 2$, $d = 4$ case, the dynamics of N^{++} in $N = 4$, $d = 1$ harmonic superspace should differ essentially, in particular due to the different dimension and harmonic U(1) charge of the $d = 1$ analytic superspace integration measure.

Other interesting problems one can try to attack within the harmonic superspace approach are the setting up of new $N = 8$, $d = 1$ SQM models by combining several analytic $N = 4$ multiplets into an irreducible off-shell $N = 8$ multiplet as well as the study of the relationship between superconformal $N = 4$, $d = 1$ models and superparticles on $\mathrm{AdS}_2 \times S^2$ as a special case of the general AdS/CFT correspondence.

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